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Decay in time of a state subject to a renormalised interaction

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Abstract. When a theory containing a divergent level shift is renormalised, all energy moments above and including the second (essentially $\langle 0|H^2|0\rangle$) continue to diverge. A mathematical model of this situation is considered and it is shown that the decay of a state in time is unambiguous and well behaved, despite the divergences. The only novelty is that the usual t^2 decay term at small times is replaced by $t^2 \log t$.

1. Introduction

Suppose that a Hamiltonian H_0 has a discrete solution $|0\rangle$, energy E_0, and a continuum of solutions $|E\rangle$ with energies E starting at a threshold $E=0$:

$$\begin{aligned} &\langle 0|0\rangle = 1, \qquad H_0|0\rangle = E_0|0\rangle \\ &\langle E|E'\rangle = \delta(E-E'), \qquad H_0|E\rangle = E|E\rangle \\ &\langle 0|E\rangle = 0. \end{aligned} \tag{1}$$

E_0 is assumed positive. The system is in state $|0\rangle$ at time $t \leqslant 0$. At $t=0$, a time-independent interaction h between $|0\rangle$ and the continuum is switched on. Without loss of generality we can assume that this has the properties:

$$\langle 0|h|0\rangle = 0, \qquad \langle E|h|E'\rangle = 0. \tag{2}$$

The state $|0\rangle$ decays away under the effect of h and this is described by the 'survival amplitude':

$$F(t) \equiv \langle 0|\exp[-i(H-E_0)t]|0\rangle \tag{3}$$

(where we set $\hbar = 1$). If M_n are the moments:

$$M_n \equiv \langle 0|(H-E_0)^n|0\rangle \tag{4}$$

where $H \equiv H_0 + h$, then, if all M_n are finite, $F(t)$ is given by expanding the exponential in (3):

$$F(t) = 1 + \sum_{n=2}^{\infty} \frac{M_n(-it)^n}{n!} \tag{5}$$

where we have used: $M_0 = 1$, $M_1 = 0$. At small times, this gives the well known quadratic

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form of initial decay (e.g. Perez (1980))

$$F(t) \approx 1 - \tfrac{1}{2} M_2 t^2 + O(t^3). \tag{6}$$

Eventually, in almost all cases of interest, the leading term in $(F(t)-1)$ becomes linear in t, corresponding to the onset of exponential decay. (There is no contradiction with the absence of a linear term in (5). For example, the form $(1+\alpha)\exp(-x\alpha t) - \alpha \exp[-x(1+\alpha)t]$ with $x>0$, $\alpha>0$ and $\alpha \ll 1$ has form (6) at small times $t \ll x^{-1}$, and becomes $\approx \exp(-x\alpha t)$ at times $t \gg x^{-1}$.)

Perez assumed that M_2 is finite. The subject of this work is the effect on $F(t)$ of the divergence of M_n for n above a certain value. As an extreme case, M_2 diverges so that all M_n in (5) diverge. From (4), M_2 can be written:

$$M_2 = \langle 0|(H-E_0)^2|0\rangle = \int_0^\infty h_E^2 \, dE \tag{7}$$

where $h_E \equiv \langle E|h|0\rangle$. This can be compared to the level shift of level $|0\rangle$ due to coupling h:

$$\delta_R(E_0) = -P \int_0^\infty \frac{h_E^2}{E-E_0} \, dE. \tag{8}$$

In a renormalised theory, this quantity initially diverges logarithmically but is rendered finite by renormalisation. This procedure will in general not remove the divergence of M_2, although it will reduce its severity from linear to logarithmic divergence. The effect of this divergence has not been studied in previous discussions of decay (Perez 1980, Fonda *et al* 1978).

One might have guessed that the divergence of all terms in series (5) would mean that no sensible form of $F(t)$ of (3) exists, so that such theories are of dubious physical relevance. We will see that this is not so, and that $F(t)$ is a well defined significant quantity even when all M_n diverge. However $(F(t)-1)$ departs from the usual quadratic form at small t, having the form $t^2 \log t$.

In § 2, we summarise some essential results of general decay theory. In § 3, we construct the explicit form of $F(t)$ for a mathematical model of h_E^2 with convergent δ_R and divergent M_2.

2. General decay theory

The amplitude $F(t)$ of (3) can be approached from two directions: the Fourier transform of an explicit function, or the solution of an integro-differential equation.

The first method (e.g. Goldberger and Watson 1964) involves the introduction of the function:

$$f(E) = i \int_0^\infty F(t) \exp i(E^+ - E_0)t \, dt \tag{9}$$

where $E^+ \equiv E + i|\varepsilon|$, and $|\varepsilon| \to 0$ after integration. From (3)

$$f(E) = \langle 0|1/(H-E^+)|0\rangle. \tag{10}$$

We retrieve $F(t)$ by inserting this in

$$F(t)=\frac{1}{2\pi i}\int_{-\infty}^{\infty} f(E)\exp[-i(E-E_0)t]\,dE. \tag{11}$$

An equivalent form to (10) is

$$f(E)=(E_0-E+\delta(E))^{-1} \tag{12}$$

where

$$\delta(E)=-\langle 0|h(H_0-E^+)^{-1}h|0\rangle=-\int [h_{E'}^2/(E'-E^+)]\,dE'. \tag{13}$$

The real part, $\delta_R(E)$, equals the principal part (and agrees with (8) when E is chosen $=E_0$) while the imaginary part, $\delta_I(E)$, $=-\pi h_E^2$.

As an alternative to (11)

$$F(t)=\frac{1}{\pi}\int_0^{\infty}\exp[-i(E-E_0)t]\,\mathrm{Im}\,f(E)\,dE \tag{14}$$

obtained by changing the contour to one enclosing the positive real axis. In general, there would be an additional sum over bound states of H in (14), but the present model (1) has none (unless $\delta_R(E_o)$ is negative and so large that $E_0+\delta_R(E_0)$ is negative, which possibility we will exclude). From (12)

$$\mathrm{Im}\,f(E)=\pi h_E^2/[(E_0+\delta_R(E)-E)^2+\pi^2 h_E^4]. \tag{15}$$

The second approach to $F(t)$ (e.g. Perez 1980) may be obtained directly from the Schrödinger equation, or from taking the Fourier transform of a version of (12):

$$f(E)\delta(E)+f(E)(E-E_0)-1=0. \tag{16}$$

It is given by

$$\frac{dF}{dt}=\int_0^t \Delta(\tau)F(t-\tau)\,d\tau \tag{17}$$

where $\Delta(t)$ is related to $\delta(E)$ as $F(t)$ is related to $f(E)$:

$$\Delta(t)\equiv\frac{1}{2\pi i}\int_{-\infty}^{\infty}\delta(E)\exp[-i(E-E_0)t]\,dE=-\int_0^{\infty}h_E^2\exp[-i(E-E_0)t]\,dE. \tag{18}$$

Integration of (17) gives the alternative form:

$$F(t)=1+\int_0^t F(t')K(t-t')\,dt' \tag{19}$$

where

$$K(\tau)=\int_0^{\tau}\Delta(t')\,dt'. \tag{20}$$

Defining the moments

$$m_k\equiv\langle 0|h(H_0-E_0)^{k-2}h|0\rangle=\int_0^{\infty}h_E^2(E-E_0)^{k-2}\,dE \tag{21}$$

then, provided these are finite, at small times, $\Delta(t)$ of (18) has an expansion like (5):

$$\Delta(t) = -\sum_{k=2} m_k \frac{(-\mathrm{i}t)^{k-2}}{(k-2)!}. \tag{22}$$

The relationship between the moments m_k and M_n is found by taking (17) at small t,

$$M_n = \sum_{k=2}^{n} m_k M_{n-k}. \tag{23}$$

Recalling that $M_0 = 1$, $M_1 = 0$, this gives $M_2 = m_2$, $M_3 = m_3$, $M_4 = m_4 + m_2^2$, etc.

Perez (1980) has discussed low-t decay in terms of moments M_n, assuming that these converge. When moments diverge, this does not mean that small-time expansions of $F(t)$ or $\Delta(t)$ do not exist, but simply that the integration on E in (9) or (18) must be done *before* expansion in powers of t, not after, these two operations not being commutable in general. As an elementary example of this, we may consider the case

$$h_E^2 = \pi^{-1} M^2 W / [(E - E_0)^2 + W^2] \tag{24}$$

with E_0 so far above threshold that the integration in (18) can be taken from $(E - E_0) = -\infty$ to $+\infty$, then

$$\Delta(t) = -M^2 \exp(-Wt) \tag{25}$$

which can be expanded at low t, despite the fact that all m_k with $k \geqslant 3$ diverge. When m_k with k up to a certain value are finite, they give the correct coefficients in the low t expansions of $F(t)$ or $\Delta(t)$. In the present example the value of $-(\mathrm{d}^2F/\mathrm{d}t^2)_{t=0}$ from (17), (25) is M^2 and this equals $m_2 = M_2$ from (21), (24).

3. Explicit solution for a mathematical model

We consider the form:

$$h_E^2 = M^2 E / (E^2 + W^2) \tag{26}$$

which has the properties that $h_E^2 \to 0$ linearly as $E \to 0$, and $h_E^2 \to 0$ like E^{-1} when $E \to \infty$. The latter property means that all M_n in (5) and m_k in (21) diverge.

The first method of solution for $F(t)$ based on (14), (15) or (11), (12) leads directly to a serious difficulty since $\delta_{\mathrm{R}}(E)$ contains a logarithmic part:

$$\delta_{\mathrm{R}}(E) = -[M^2/(W^2 + E^2)][\tfrac{1}{2}\pi W - E \log(E/W)] \tag{27}$$

so the integrations (14), (11) cannot be readily performed. It might be thought to be a reasonable procedure to drop the logarithmic term since this is negligible compared to other terms as $E \to 0$ and $E \to \infty$. Unfortunately this leads to unacceptable results: different forms come from (14) and (11), also $F(0) \neq 1$ and $(\mathrm{d}F/\mathrm{d}t)_{t=0} \neq 0$.

Thus we solve with the second method based on (17) or (19), the key quantity being (from (26) and (18)):

$$\Delta(t) = -M^2 \exp(\mathrm{i}E_0 t) \int_0^\infty \frac{E \exp(-\mathrm{i}Et)}{E^2 + W^2} \mathrm{d}E \tag{28}$$

$$= \exp(\mathrm{i}E_0 t) M^2 \left(\tfrac{1}{2}\mathrm{i}\pi \exp(-Wt) - \mathrm{P} \int_0^\infty \frac{\varepsilon \exp(-\varepsilon t)}{\varepsilon^2 - W^2} \mathrm{d}\varepsilon \right), \tag{29}$$

where P means principal part. In the appendix, we give relevant properties of the integral. From now on, we assume that the bandwidth W is much greater than escape energy E_0 (implying that $\delta_R(E_0)$ of (27) is negative) and consider the time regimes $t \ll W^{-1}$, $W^{-1} \ll t \ll E_0^{-1}$, $E_0^{-1} \ll t$.

From the appendix, for $t \ll W^{-1}$

$$\Delta(t) = M^2(\tfrac{1}{2}i\pi(1 - Wt) + \log Wt + \gamma + O((Wt)^2 \log Wt) \tag{30a}$$

$$\int_0^t K(t')\,dt' = M^2 t^2((\tfrac{1}{2}\log Wt + \tfrac{1}{2}\gamma - \tfrac{3}{4} + \tfrac{1}{4}i\pi + O((Wt)\log Wt))). \tag{30b}$$

The last quantity is of interest since, from (19), while $F(t)$ is still close to 1, it gives the correction $F(t) - 1$. We thus see that the leading correction to $F(t)$ at small t is not of the familiar t^2-form but rather $t^2 \log t$.

For $W^{-1} \ll t \ll E_0^{-1}$

$$\Delta(t) = M^2((Wt)^{-2} + 6(Wt)^{-4} + O((Wt)^{-6}))) \tag{31a}$$

$$K(t) = \frac{M^2}{W}(\tfrac{1}{2}i\pi - (Wt)^{-1} - 2Wt^{-3} + O((Wt)^{-5})) \tag{31b}$$

$$\int_0^t K(t')\,dt' = \frac{M^2}{W^2}(\tfrac{1}{2}i\pi Wt - \log Wt - \tfrac{1}{2}i\pi - \gamma + O((Wt)^{-2})). \tag{31c}$$

Thus the correction changes to a linear one when t increases beyond W^{-1}. This remains small as t increases towards E_0^{-1} provided $M^2 \ll E_0 W$ i.e. $|\delta_R(E_0)| \ll E_0$. This condition has already been assumed above (15).

For $t \gg E_0^{-1} \gg W^{-1}$ (from results in the appendix):

$$K(t) = -i\delta(E_0) - \frac{iM^2 E_0 \exp(iE_0 t)}{W^2 (E_0 t)^2}(1 + O((E_0 t)^2)) \tag{32a}$$

$$\int_0^t K(t')\,dt' = -it\delta(E_0) + \frac{M^2}{W^2}[-1 + \log(E_0/W) - i\pi] + \frac{iM^2 \exp(iE_0 t)}{W^2 E_0 t} + O((E_0 t)^{-2}). \tag{32b}$$

The leading term in the last quantity is the linear one, so from (19), while $F \approx 1$, the leading correction to $F(t)$ is the same linear term:

$$F(t) \approx 1 - it\delta(E_0) \tag{33}$$

with corrections of order $(|\delta(E_0)|/W)\log(E_0/W) \ll |\delta(E_0)|/E_0$ and $(E_0 t)^{-1}$.

For $T \gtrsim \delta^{-1}(E_0)$, corrections to $F(t) \approx 1$ are large: a first approximation to $F(t)$ is obtained from (19) on noting from (32a) the near constancy of $K(t)$ at the value $-i\delta(E_0)$ for $t \gtrsim E_0^{-1}$:

$$F(t) \approx 1 - i\delta(E_0)\int_0^{t-T} F(t')\,dt' + F(t)\int_0^T K(t')\,dt' \tag{34}$$

where T is a time such that $E_0^{-1} \ll T \ll \delta(E_0)^{-1}$. From (32), to within a small term $\ll |\delta(E_0)|/E_0$ one can set $T = 0$ in which case the solution is:

$$F(t) = \exp(-i\delta(E_0)t. \tag{35}$$

An improvement comes from the correction term in (32*b*) which leads to:

$$F(t) \approx 1 - i\delta(E_0) \int_0^{t-T} F(t')\,dt' + F(t) \int_0^T K(t')\,dt' - \frac{iM^2}{W^2 E_0} \int_T^t \frac{F(t-t')\exp(iE_0 t')}{t'^2}\,dt'. \tag{36}$$

On trying the solution for $t \gg T$:

$$F(t) = \exp(-i\delta(E_0)t) + \frac{M^2 \exp(iE_0 t)}{W^2 (E_0 t)^2} \tag{37}$$

one finds that it solves (36) to within surviving terms of relative order $(\delta(E_0)/E_0)$. The presence of the long-time correction to exponential decay has, of course, been demonstrated before (e.g. Goldberger and Watson 1964, Knight 1977, Fonda *et al* 1978).

Finally, for comparison, we briefly consider a completely convergent model as an alternative to (26):

$$h_E^2 = M^2 E W^{-2} \exp(-E/W). \tag{38}$$

This has the properties that $m_2 = M_2 = M^2$, and all higher moments are finite. From (18):

$$\Delta(t) = -M^2(iE_0 t)/(1 + iWt)^2. \tag{39}$$

For $t \ll W^{-1}$, one readily finds:

$$F(t) = 1 - \tfrac{1}{2}M^2 t^2 + O(t^3) \tag{40}$$

and for $W^{-1} \ll t \ll E_0^{-1}$:

$$F(t) = 1 + M^2 W^{-2}(iWt - \tfrac{1}{2}i\pi - \log Wt + O(Wt)^{-1}). \tag{41}$$

The condition that F remains ≈ 1 as $t \to E_0^{-1}$ is, as before, $|\delta(E_0)| \ll E_0$. For $t \gg E_0^{-1}$, we find (32*a*) again except that $\delta(E_0)$ is now appropriate to model (38), viz for $W \gg E_0$:

$$\delta(E_0) = -M^2 W^{-1}(1 + i\pi E_0 W^{-1}). \tag{42}$$

All further results (33)–(37) of the previous model apply in the present case. Thus the only essential effect of exchanging convergent and divergent models (both satisfying the conditions $|\delta(E_0)| \ll E_0$, $h_E^2 \propto E$ as $E \to 0$) is at very small times $t \ll W^{-1}$ where they give t^2 and $t^2 \log t$ corrections to $F(t) = 1$. This finding agrees with the principle that short-time decay depends only on the large-energy behaviour of h_E^2.

4. Conclusions

The leading term in the time decay of a state $|0\rangle$ is normally (Perez 1980)

$$F(t) = 1 - \tfrac{1}{2}M_2 t^2 \tag{43}$$

where M_2 is the second moment:

$$M_2 = \langle 0|H^2|0\rangle - \langle 0|H|0\rangle^2. \tag{44}$$

Perez assumed M_2 to be finite. One might have guessed that a divergent value of $\langle 0|H^2|0\rangle$ could have drastic consequences for normal decay theory, and possibly lead to unacceptable paradoxes. We have shown by explicit consideration of a simple mathematical model that this is not the case. All results of normal theory, such as the 'golden-rule' results (33), (35) apply, the only exception being that the t^2-term in (43) becomes $t^2 \log t$ for $t \ll W^{-1}$. This difference is of only academic interest since such times $t \ll W^{-1}$ are so small as to be beyond experimental resolution.

Appendix. Explicit forms implied by model (26)

The integral in expression (28) for $\Delta(t)$ can be written thus:

$$\mathrm{P}\int_0^\infty \frac{\varepsilon\,\mathrm{e}^{-\varepsilon}}{\varepsilon^2-(Wt)^2}\,\mathrm{d}\varepsilon = \tfrac{1}{2}[\exp(Wt)E_1(Wt)+\exp(-Wt)E_1(-Wt)] \tag{A1}$$

where:

$$E_1(x) \equiv \int_x^\infty \exp(-y)/y\,\mathrm{d}y \tag{A2}$$

and the principal part is taken if $x<0$. For small $|x| \ll 1$, $E_1(\pm|x|)$ is approximated by

$$E_1(|x|) = -\gamma - \log|x| + |x| - \tfrac{1}{4}|x|^2 + \mathrm{O}(|x|^3) \tag{A3}$$

$$E_1(-|x|) = -\gamma - \log|x| - |x| - \tfrac{1}{4}|x|^2 + \mathrm{O}(|x|^3)$$

while for large $|x| \gg 1$:

$$E_1(|x|) = \mathrm{e}^{-|x|}\left(\frac{1}{|x|} - \frac{1}{|x|^2} + \frac{2}{|x|^3}\dots\right) \tag{A4}$$

$$E_1(-|x|) = -\mathrm{e}^{|x|}\left(\frac{1}{|x|} + \frac{1}{|x|^2} + \frac{2}{|x|^3}\dots\right)$$

In terms of the same quantities $E_1(\pm Wt)$, $K(t)$ of (20) is, for $t \ll E_0^{-1}$:

$$K(t) = \tfrac{1}{2}M^2W^{-1}[\mathrm{i}\pi[1-\exp(-Wt)] + \exp(-Wt)E_1(-Wt) - \exp(Wt)E_1(Wt)] \tag{A5}$$

and its integral is:

$$\int_0^t K(t')\,\mathrm{d}t' = W^{-2}[\tfrac{1}{2}\mathrm{i}\pi M^2(\exp(-Wt) - 1 + Wt) + \operatorname{Re}\Delta(t) - M^2(\gamma + \log Wt)]. \tag{A6}$$

For general t, these are:

$$K(t) = -\mathrm{i}\delta(E_0) + \tfrac{1}{2}\mathrm{i}M^2\,\mathrm{e}^{\mathrm{i}E_0t}\left(\frac{\mathrm{e}^{-Wt}}{E_0+\mathrm{i}W}(-\mathrm{i}\pi + E_1(-Wt) + \frac{\mathrm{e}^{Wt}E_1(Wt)}{E_0-\mathrm{i}W}\right)$$
$$+\frac{\mathrm{i}M^2E_0}{E_0^2+W^2}\,\xi(E_0t) \tag{A7}$$

$$\int_0^t K(t')\,\mathrm{d}t' = -\mathrm{i}t\delta(E_0) + \tfrac{1}{2}M^2\left[\frac{2}{(E_0^2+W^2)^2}\{-(E_0^2-W^2)(\log E_0/W - \mathrm{i}\pi)\right.$$
$$+\pi E_0W - (E_0^2+W^2)\} + \mathrm{e}^{\mathrm{i}E_0t}\left\{\frac{\exp(-Wt)E_1(-Wt)}{(E_0+\mathrm{i}W)^2} + \frac{\exp(Wt)E_1(Wt)}{(E_0-\mathrm{i}W)^2}\right.$$
$$\left.\left.-\frac{2}{E_0^2+W^2} - \frac{\mathrm{i}\pi\exp(Wt)}{(E_0+\mathrm{i}W)^2}\right\} - 2\left(\frac{E_0^2-W^2}{(E_0^2+W^2)^2} + \frac{\mathrm{i}E_0t}{E_0^2+W^2}\right)\xi(E_0t)\right] \tag{A8}$$

where

$$\xi(x) \equiv \int_x^\infty \frac{\exp(\mathrm{i}y)}{y}\,\mathrm{d}y \tag{A9}$$

and we have used the fact that $\xi(-|x|) = \xi(|x|) - \mathrm{i}\pi$. This has the form for large x:

$$\xi(x) = \mathrm{e}^{\mathrm{i}x}(\mathrm{i}/x + 1/x^2 + \ldots).$$

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